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The $m = 1$ amplituhedron and cyclic hyperplane arrangements

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Abstract. The *(tree) amplituhedron* $\mathcal{A}_{n,k,m}$ is the image in the Grassmannian $\mathrm{Gr}_{k,k+m}$ of the totally nonnegative part of $\mathrm{Gr}_{k,n}$, under a (map induced by a) linear map which is totally positive. It was introduced by Arkani-Hamed and Trnka in 2013 in order to give a geometric basis for the computation of scattering amplitudes in $\mathcal{N} = 4$ supersymmetric Yang-Mills theory. When $k + m = n$, the amplituhedron is isomorphic to the totally nonnegative Grassmannian, and when $k = 1$, the amplituhedron is a cyclic polytope. While the case $m = 4$ is most relevant to physics, the amplituhedron is an interesting mathematical object for any m. We study it in the case $m = 1$. We start by taking an orthogonal point of view and define a related "B-amplituhedron" $\mathcal{B}_{n,k,m}$, which we show is isomorphic to $\mathcal{A}_{n,k,m}$. We use this reformulation to describe the amplituhedron in terms of sign variation. We then give a cell decomposition of the amplituhedron $\mathcal{A}_{n,k,1}$ using the images of a collection of distinguished cells of the totally nonnegative Grassmannian. We also show that $\mathcal{A}_{n,k,1}$ can be identified with the complex of bounded faces of a *cyclic hyperplane arrangement*. We deduce that $\mathcal{A}_{n,k,1}$ is homeomorphic to a ball.

Résumé. L'amplituèdre $\mathcal{A}_{n,k,m}$ est l'image dans la grassmannienne $\mathrm{Gr}_{k,k+m}$ de la partie totalement non négative de $\mathrm{Gr}_{k,n}$ par une (application induite par une) application linéaire totalement positive. Il a introduit par Arkani-Hamed et Trnka en 2013 afin de fournir une base géométrique pour le calcul des amplitudes de diffusion dans la théorie supersymétrique $\mathcal{N} = 4$ de Yang-Mills. Dans le cas $k + m = n$, l'amplituèdre est isomorphe à la grassmannienne totalement non négative, et dans le cas $k = 1$, l'amplituèdre est un polytope cyclique. Bien que le cas $m = 4$ est le plus pertinent pour la physique, l'amplituèdre est un objet d'intérêt mathématique pour tous m. Nous l'étudions dans le cas $m = 1$. Nous commençons par prendre un point de vue orthogonal et définissons $\mathcal{B}_{n,k,m}$, le «B-amplituèdre», que nous montrons est isomorphe à $\mathcal{A}_{n,k,m}$. Nous utilisons cette reformulation pour décrire l'amplituèdre en termes de variations de signe. Nous fournissons ensuite une décomposition cellulaire de l'amplituèdre $\mathcal{A}_{n,k,1}$, en utilisant les images d'une collection de cellules distinguées de la grassmannienne totalement non négative. Nous montrons également qu'on peut identifier $\mathcal{A}_{n,k,1}$ au complexe de faces bornées d'un *arrangement cyclique d'hyperplans*. Nous déduisons que $\mathcal{A}_{n,k,1}$ est homéomorphe à une boule.

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1 Introduction

The totally nonnegative Grassmannian $\mathrm{Gr}_{k,n}^{\geq 0}$ is the subset of the real Grassmannian $\mathrm{Gr}_{k,n}$ consisting of points with all Plücker coordinates nonnegative. Following seminal work of Lusztig [15], as well as by Fomin and Zelevinsky [7], Postnikov initiated the combinatorial study of $\mathrm{Gr}_{k,n}^{\geq 0}$ and its cell decomposition [18]. Since then the totally nonnegative Grassmannian has found applications in diverse contexts such as mirror symmetry [16], soliton solutions to the KP equation [13], and scattering amplitudes for $\mathcal{N} = 4$ supersymmetric Yang-Mills theory [2].

Building on [2], Arkani-Hamed and Trnka [1] recently introduced a beautiful new mathematical object called the *(tree) amplituhedron*, which is the image of the totally nonnegative Grassmannian under a particular map.

Definition 1.1. Let Z be a $(k+m) \times n$ real matrix whose maximal minors are all positive, where $m \geq 0$ is fixed with $k + m \leq n$. Then it induces a map

$$\tilde{Z} : \mathrm{Gr}_{k,n}^{\geq 0} \to \mathrm{Gr}_{k,k+m}$$

defined by

$$\tilde{Z}(\langle v_1, \ldots, v_k \rangle) := \langle Z(v_1), \ldots, Z(v_k) \rangle,$$

where $\langle v_1, \ldots, v_k \rangle$ is an element of $\mathrm{Gr}_{k,n}^{\geq 0}$ written as the span of k basis vectors. (The fact that Z has positive maximal minors ensures that $\tilde{Z}$ is well defined.) The *(tree) amplituhedron* $\mathcal{A}_{n,k,m}(Z)$ is defined to be the image $\tilde{Z}(\mathrm{Gr}_{k,n}^{\geq 0})$ inside $\mathrm{Gr}_{k,k+m}$.

In special cases the amplituhedron recovers familiar objects. If Z is a square matrix, i.e. $k + m = n$, then $\mathcal{A}_{n,k,m}(Z)$ is isomorphic to the totally nonnegative Grassmannian. If $k = 1$, then $\mathcal{A}_{n,1,m}(Z)$ is a *cyclic polytope* in projective space [25].

While the amplituhedron $\mathcal{A}_{n,k,m}(Z)$ is an interesting mathematical object for any m, the case of immediate relevance to physics is $m = 4$. In this case, it provides a geometric basis for the computation of *scattering amplitudes* in $\mathcal{N} = 4$ supersymmetric Yang-Mills theory. These amplitudes are complex numbers related to the probability of observing a certain scattering process of n particles. It is expected that such amplitudes can be expressed (modulo higher-order terms) as an integral over the amplituhedron $\mathcal{A}_{n,k,4}(Z)$. This statement would follow from the conjecture of Arkani-Hamed and Trnka [1] that the images of a certain collection of $4k$-dimensional cells of $\mathrm{Gr}_{k,n}^{\geq 0}$ provide a "triangulation" of the amplituhedron $\mathcal{A}_{n,k,4}(Z)$. More specifically, the BCFW recurrence [3, 4] provides one way to compute scattering amplitudes. Translated into the Grassmannian

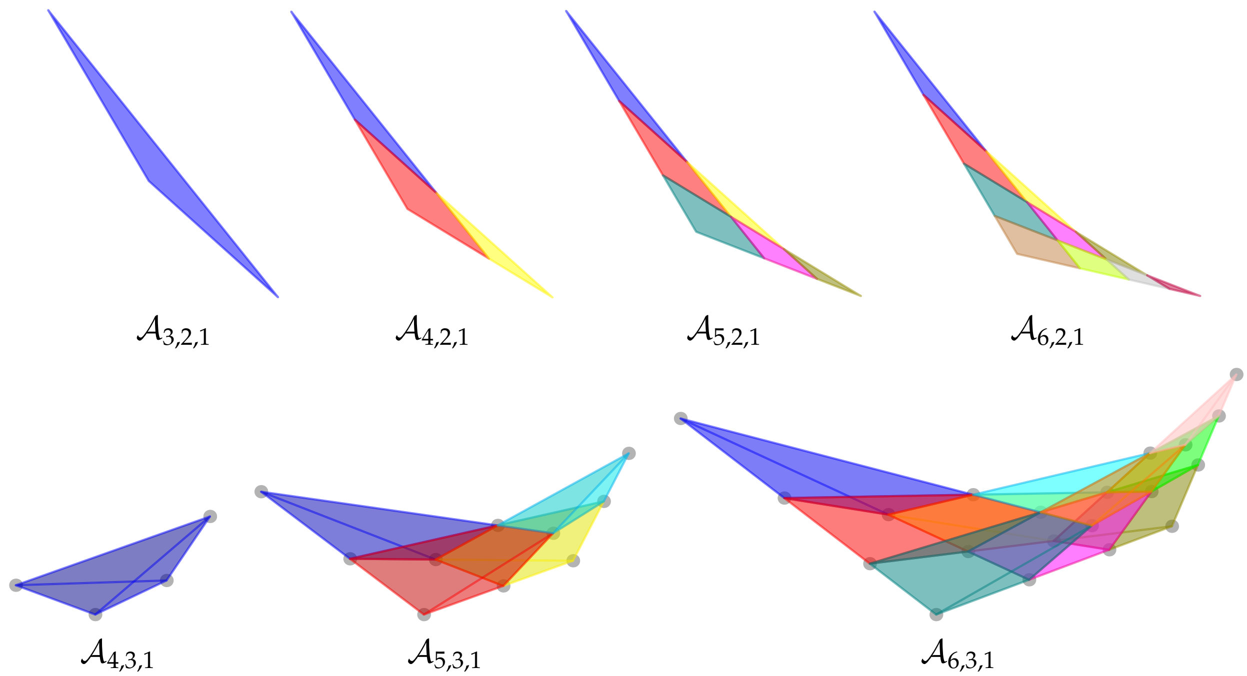


Figure 1: The amplituhedron $\mathcal{A}_{n,k,1}(Z)$ as the complex of bounded faces of a cyclic hyperplane arrangement of n hyperplanes in $\mathbb{R}^k$, for $k = 2, 3$ and $n \leq 6$.

formulation of [2], the terms in the BCFW recurrence can be identified with a collection of $4k$-dimensional cells in $\mathrm{Gr}_{k,n}^{\geq 0}$. If the images of these *BCFW cells* in $\mathcal{A}_{n,k,4}(Z)$ fit together in a nice way, then we can combine the contributions from each term into a single integral over $\mathcal{A}_{n,k,4}(Z)$.

We study the amplituhedron $\mathcal{A}_{n,k,1}(Z)$ for $m = 1$. We find that this object is already interesting and non-trivial. Since $\mathcal{A}_{n,k,1}(Z) \subseteq \mathrm{Gr}_{k,k+1}$, it is convenient to take orthogonal complements and work with lines rather than k-planes in $\mathbb{R}^{k+1}$. This leads us to define a related "B-amplituhedron"

$$\mathcal{B}_{n,k,m}(W) := \{V^\perp \cap W : V \in \mathrm{Gr}_{k,n}^{\geq 0}\} \subseteq \mathrm{Gr}_m(W),$$

which is homeomorphic to $\mathcal{A}_{n,k,m}(Z)$, where W is the subspace of $\mathbb{R}^n$ spanned by the rows of Z (Section 3). We then describe $\mathcal{B}_{n,k,m}(W)$ explicitly in terms of sign variation.

Using this description, we show that the $m = 1$ amplituhedron is triangulated by the images of certain k-dimensional cells of $\mathrm{Gr}_{k,n}^{\geq 0}$, which come from an $m = 1$ analogue of the BCFW recursion. More specifically, we prove that $\mathcal{A}_{n,k,1}(Z)$ is homeomorphic to a k-dimensional subcomplex of the totally nonnegative Grassmannian $\mathrm{Gr}_{k,n}^{\geq 0}$ (Section 4).

See Figure 2 for $\mathcal{A}_{4,2,1}(Z)$ as a subcomplex of $\mathrm{Gr}_{2,4}^{\geq 0}$. We also show that $\mathcal{A}_{n,k,1}(Z)$ can be identified with the complex of bounded faces of a certain hyperplane arrangement of n hyperplanes in $\mathbb{R}^k$, called a *cyclic hyperplane arrangement* (Section 5). Along the way to proving this result, we determine which sign vectors label the bounded and unbounded faces of such an arrangement.

It is known that the totally nonnegative Grassmannian has a remarkably simple topology: it is contractible with boundary a sphere [21], and its poset of cells is Eulerian [26]. While there are not yet any general results in this direction, calculations of Euler characteristics [9] indicate that the amplituhedron $\mathcal{A}_{n,k,m}(Z)$ is likely also topologically very nice. Our description of $\mathcal{A}_{n,k,1}(Z)$ as the complex of bounded faces of a hyperplane arrangement, together with a result of Dong [6], implies that the $m = 1$ amplituhedron is homeomorphic to a closed ball (Corollary 5.6).

Further results and the proofs of all results stated here appear in our preprint [12].

2 Background on the Grassmannian and sign variation

The *(real) Grassmannian* $\mathrm{Gr}_{k,n}$ is the space of all k-dimensional subspaces of $\mathbb{R}^n$, for $0 \leq k \leq n$. An element of $\mathrm{Gr}_{k,n}$ can be viewed as a $k \times n$ matrix of rank k, modulo row operations. Let $[n]$ denote $\{1, \ldots, n\}$, and $\binom{[n]}{k}$ the set of all k-element subsets of $[n]$. Given $V \in \mathrm{Gr}_{k,n}$ represented by a $k \times n$ matrix A, for $I \in \binom{[n]}{k}$ we let $\Delta_I(V)$ be the maximal minor of A located in the column set I. The $\Delta_I(V)$ do not depend on our choice of matrix A (up to simultaneous rescaling by a nonzero constant), and give projective coordinates on $\mathrm{Gr}_{k,n}$ called *Plücker coordinates*.

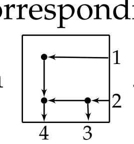
Definition 2.1 ([18, Section 3])**.** We say that $V \in \mathrm{Gr}_{k,n}$ is *totally nonnegative* if $\Delta_I(V) \geq 0$ for all $I \in \binom{[n]}{k}$, and *totally positive* if $\Delta_I(V) > 0$ for all $I \in \binom{[n]}{k}$. The set of all totally nonnegative $V \in \mathrm{Gr}_{k,n}$ is the *totally nonnegative Grassmannian* $\mathrm{Gr}_{k,n}^{\geq 0}$, and the set of all totally positive V is the *totally positive Grassmannian* $\mathrm{Gr}_{k,n}^{>0}$. For $M \subseteq \binom{[n]}{k}$, the *positroid cell* S_M is the set of $V \in \mathrm{Gr}_{k,n}^{\geq 0}$ with the prescribed collection of Plücker coordinates strictly positive (i.e. $\Delta_I(V) > 0$ for all $I \in M$), and the remaining Plücker coordinates equal to zero (i.e. $\Delta_J(V) = 0$ for all $J \in \binom{[n]}{k} \setminus M$). We call M a *positroid* if S_M is nonempty. We let $Q_{k,n}$ denote the poset on the cells of $\mathrm{Gr}_{k,n}^{\geq 0}$ defined by $S_M \leq S_{M'}$ if and only if $S_M \subseteq \overline{S_{M'}}$.

In [18], Postnikov defined several families of combinatorial objects which are in bijection with positroids. We will work with one particular set of objects, called Ⅎ-diagrams.

Definition 2.2. Fix k and n, and let λ be a partition whose Young diagram is contained in the $k \times (n-k)$ rectangle. A *Ⅎ-diagram* D of shape λ and type (k, n) is a filling of the Young diagram of λ with the symbols 0 and $+$, such that there is no 0 which has a $+$ above it in the same column and a $+$ to its left in the same row.

To each Ꞁ-diagram D of type (k,n), we associate a positroid $M(D) \subseteq \binom{[n]}{k}$ as follows. We delete the 0's of D and replace each $+$ with a hook, which extends east and south to border of the Young diagram. We label the edges along the southeast border of the Young diagram by $1,\dots,n$, which we regard as boundary vertices, and let $I \in \binom{[n]}{k}$ be the labels of the vertical edges. Then $M(D)$ is the set of $J \in \binom{[n]}{k}$ such that there exists a *flow* from I to J, i.e. a family of k nonintersecting paths each joining a vertex in I to a vertex in J, travelling only left and down along the edges of the graph. Postnikov [18, Theorem 17.1] showed that $D \mapsto M(D)$ is a bijection from Ꞁ-diagrams of type (k,n) to positroids which index the cells of $\mathrm{Gr}_{k,n}^{\geq 0}$, where $\dim(S_{M(D)})$ is the number of $+$'s in D. We partially order Ꞁ-diagrams according to the partial order on positroid cells.

Figure 2 shows the poset $Q_{2,4}$, with elements labeled by their corresponding Ꞁ-diagram.

Example 2.3. Let $V \in \mathrm{Gr}_{2,4}^{\geq 0}$ be represented by the matrix $\begin{bmatrix} 1 & 0 & 0 & -1 \\ -1 & 2 & 1 & 3 \end{bmatrix}$. Then $V \in S_M$, where $M := \{\{1,2\},\{1,3\},\{1,4\},\{2,4\},\{3,4\}\}$. The Ꞁ-diagram corresponding to M is $D := \begin{array}{|c|c|}\hline + & 0 \\ \hline + & + \\ \hline\end{array}$. We can read off M as in Definition 2.2 from the graph 

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Definition 2.4. Given $v \in \mathbb{R}^n$, let $\mathrm{var}(v)$ be the number of times v changes sign, viewed as a sequence of n numbers and ignoring any zeros. (We set $\mathrm{var}(0) := -1$.) We also let

$$\overline{\mathrm{var}}(v) := \max\{\mathrm{var}(w) : w \in \mathbb{R}^n \text{ such that } w_i = v_i \text{ for all } i \in [n] \text{ with } v_i \neq 0\},$$

i.e. $\overline{\mathrm{var}}(v)$ is the maximum number of times v changes sign after we choose a sign for each zero. For example, if $v := (4,-1,0,-2) \in \mathbb{R}^4$, then $\mathrm{var}(v) = 1$ and $\overline{\mathrm{var}}(v) = 3$.

A result of Gantmakher and Krein characterizes totally nonnegative and totally positive subspaces in terms of sign variation.

Theorem 2.5 ([10, Theorems V.3, V.7, V.1, V.6]). *Let $V \in \mathrm{Gr}_{k,n}$.*
(i) $V \in \mathrm{Gr}_{k,n}^{\geq 0} \iff \mathrm{var}(v) < k$ for all $v \in V \iff \overline{\mathrm{var}}(w) \geq k$ for all $w \in V^\perp \setminus \{0\}$.
(ii) $V \in \mathrm{Gr}_{k,n}^{>0} \iff \overline{\mathrm{var}}(v) < k$ for all $v \in V \setminus \{0\} \iff \mathrm{var}(w) \geq k$ for all $w \in V^\perp \setminus \{0\}$.

3 A complementary view of the amplituhedron

The amplituhedron $\mathcal{A}_{n,k,m}(Z)$ is a subset of $\mathrm{Gr}_{k,k+m}$. Since we are considering the case $m = 1$, it will be convenient for us to take orthogonal complements and work with subspaces of dimension m, rather than codimension m. To that end, we define an object $\mathcal{B}_{n,k,m}$, which we show is homeomorphic to $\mathcal{A}_{n,k,m}$. (We emphasize that $\mathcal{B}_{n,k,m}$ is merely an equivalent way of defining the amplituhedron, not a type B analogue of it.)

Definition 3.1. Given $W \in \mathrm{Gr}_{k+m,n}^{>0}$, let $\mathrm{Gr}_m(W)$ denote the set of $X \in \mathrm{Gr}_{m,n}$ with $X \subseteq W$. Define

$$\mathcal{B}_{n,k,m}(W) := \{V^\perp \cap W : V \in \mathrm{Gr}_{k,n}^{\geq 0}\} \subseteq \mathrm{Gr}_m(W),$$

where Theorem 2.5 implies that $\dim(V^\perp \cap W) = m$ for $V \in \mathrm{Gr}_{k,n}^{\geq 0}$. We remark that Lam used a similar construction to define *universal amplituhedron varieties* [14, Section 18].

Proposition 3.2. *Suppose that Z is a $(k+m) \times n$ matrix ($n \geq k+m$) with positive maximal minors, and $W \in \mathrm{Gr}_{k+m,n}^{>0}$ is the row span of Z. Then there is a homeomorphism from $\mathcal{A}_{n,k,m}(Z)$ to $\mathcal{B}_{n,k,m}(W)$, which sends $\tilde{Z}(V)$ to $V^\perp \cap W$ for all $V \in \mathrm{Gr}_{k,n}^{\geq 0}$.*

In the case $m = 1$, we can describe $\mathcal{B}_{n,k,1}(W)$ explicitly using sign variation. Note that $\mathrm{Gr}_1(W) = \mathbb{P}(W)$, projective space of lines in W.

Lemma 3.3. *For $W \in \mathrm{Gr}_{k+1,n}^{>0}$, we have $\mathcal{B}_{n,k,1}(W) = \{w \in \mathbb{P}(W) : \overline{\mathrm{var}}(w) = k\} \subseteq \mathbb{P}(W)$.*

In general, we have the containment

$$\mathcal{B}_{n,k,m}(W) \subseteq \{X \in \mathrm{Gr}_m(W) : k \leq \overline{\mathrm{var}}(v) \leq k+m-1 \text{ for all } v \in X \setminus \{0\}\},$$

which follows directly from Theorem 2.5. It is an important open problem to determine if equality holds. If it does, we can apply results of [11] to describe $\mathcal{B}_{n,k,m}(W)$ explicitly in terms of the Plücker coordinates of elements of $\mathrm{Gr}_m(W)$; see [12, Section 3.3].

4 $\mathcal{A}_{n,k,1}$ as a subcomplex of $\mathrm{Gr}_{k,n}^{\geq 0}$

In this section we show that the $m = 1$ amplituhedron is isomorphic to an induced subcomplex of cells of the totally nonnegative Grassmannian. We will work with our reformulation of the amplituhedron, $\mathcal{B}_{n,k,1}(W)$. We begin by defining a decomposition of $\mathcal{B}_{n,k,1}(W)$, whose pieces are indexed by sign vectors with a natural partial order.

Definition 4.1. Let $\overline{\mathrm{Sign}}_{n,k,1}$ denote the set of nonzero sign vectors $\sigma \in \{0,+,-\}^n$ with $\overline{\mathrm{var}}(\sigma) = k$, such that if $i \in [n]$ indexes the first nonzero component of σ, then $\sigma_i = (-1)^{i-1}$. We define a partial order on $\{0,+,-\}^n$, such that $\sigma \leq \tau$ if and only if $\sigma_i = \tau_i$ for all $i \in [n]$ such that $\sigma_i \neq 0$. (Equivalently, $\sigma \leq \tau$ if and only if we can obtain σ from τ by setting some components to 0.) This gives a partial order on $\overline{\mathrm{Sign}}_{n,k,1}$.

For $\sigma \in \overline{\mathrm{Sign}}_{n,k,1}$, we define $\mathcal{B}_\sigma(W) := \{w \in W \setminus \{0\} : \mathrm{sign}(w) = \pm\sigma\}$. Note that by Lemma 3.3, we indeed have $\mathcal{B}_{n,k,1}(W) = \bigsqcup_{\sigma \in \overline{\mathrm{Sign}}_{n,k,1}} \mathcal{B}_\sigma(W)$. We partially order the terms of this decomposition according to the poset structure on $\overline{\mathrm{Sign}}_{n,k,1}$. (We will show in Theorem 4.4 that $\sigma \leq \tau$ if and only if $\mathcal{B}_\sigma(W) \subseteq \overline{\mathcal{B}_\tau(W)}$.)

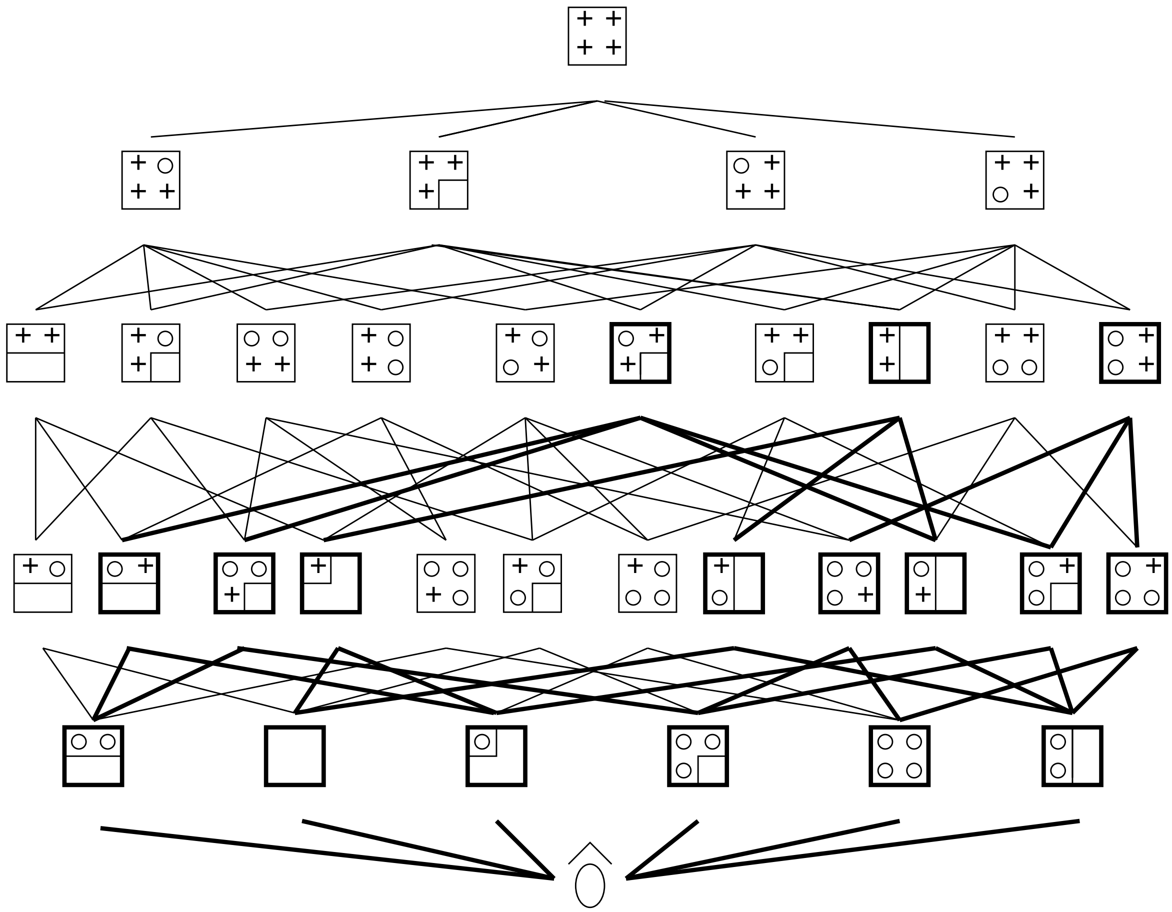

Figure 2: The poset $Q_{2,4}$ of cells of $\mathrm{Gr}_{2,4}^{\geq 0}$, where each cell is identified with the corresponding Ⅎ-diagram. The bold edges indicate the subcomplex (an induced subposet) which gets identified with the amplituhedron $\mathcal{A}_{4,2,1}(Z)$.

In order to identify $\mathcal{B}_{n,k,1}(W)$ with a subcomplex of $\mathrm{Gr}_{k,n}^{\geq 0}$, we associate to each $\sigma \in \overline{\mathrm{Sign}_{n,k,1}}$ a Ⅎ-diagram of type (k,n), as follows.

Definition 4.2. Let $\overline{\mathcal{D}_{n,k,1}}$ be the set of Ⅎ-diagrams of type (k,n) whose rows each have at most one $+$, and each $+$ appears at the right end of its row. We regard $\overline{\mathcal{D}_{n,k,1}}$ as an induced subposet of $Q_{k,n}$, in which it is an order ideal (i.e. downset); see Figure 2.

For $D \in \overline{\mathcal{D}_{n,k,1}}$, we define a sign vector $\sigma(D) \in \{0,+,-\}^n$ recursively as follows. First we label the edges of the southeast border of D by $1,\ldots,n$ from northeast to southwest. Then we set $\sigma(D)_1 := +$, and for $i = 1,\ldots,n-1$, we let $\sigma(D)_{i+1} := \sigma(D)_i$ if i labels a horizontal step of the southeast border of D, and $\sigma(D)_{i+1} := -\sigma(D)_i$ if i labels a vertical step. Finally, for each $j \in [n]$ which labels a vertical step of the southeast border of D

whose row contains no $+$, we set $\sigma(D)_j$ to zero.

For example, if $D = \begin{array}{|c|c|c|c|c|}\hline 0&0&0&0&+\\ \hline 0&0 \\ \cline{1-2}\end{array} \in \overline{\mathcal{D}_{7,2,1}}$, we have $\sigma(D) = (+,-,-,-,0,+,+)$. We remark that we can obtain the ⅃-diagrams in $\overline{\mathcal{D}_{n,k,1}}$ from an $m = 1$ analogue of the BCFW recursion; see [12, Section 4].

Lemma 4.3. *The map $D \mapsto \sigma(D)$ is a poset isomorphism from $\overline{\mathcal{D}_{n,k,1}}$ to $\overline{\mathrm{Sign}_{n,k,1}}$.*

Theorem 4.4. *Define $\mathcal{S} := \sqcup_{D\in\overline{\mathcal{D}_{n,k,1}}} S_{M(D)}$, which is a subcomplex of the cells of $\mathrm{Gr}_{k,n}^{\geq 0}$. Then the map*

$$\mathcal{S} \to \mathcal{B}_{n,k,1}(W), \quad V \mapsto V^{\perp} \cap W$$

is a homeomorphism, which induces a poset isomorphism on the decompositions of $\mathcal{S}$ and $\mathcal{B}_{n,k,1}(W)$. In particular, $\mathcal{B}_{n,k,1}(W)$ is stratified by $\overline{\mathrm{Sign}_{n,k,1}}$, and $\overline{\mathcal{B}_{\sigma}(W)} = \bigsqcup_{\tau\leq\sigma} \mathcal{B}_{\tau}(W)$ for all $\sigma \in \overline{\mathrm{Sign}_{n,k,1}}$. The codimension of $\mathcal{B}_{\sigma}(W)$ equals the number of zero components of σ.

For example, the poset of cells of $\mathcal{B}_{4,2,1}(W)$ is highlighted in bold in Figure 2. The 3 top-dimensional cells correspond to the elements $(+,-,-,+)$, $(+,+,-,+)$, $(+,-,+,+)$ of $\overline{\mathrm{Sign}_{4,2,1}}$ with no zero components.

5 $\mathcal{A}_{n,k,1}$ as the complex of bounded faces of a cyclic hyperplane arrangement

We show that the $m = 1$ amplituhedron $\mathcal{B}_{n,k,1}(W)$ (or $\mathcal{A}_{n,k,1}(Z)$) is homeomorphic to the complex of bounded faces of a *cyclic hyperplane arrangement* of n hyperplanes in $\mathbb{R}^k$. It then follows from a result of Dong [6] that it is homeomorphic to a ball. This story is somewhat analogous to that of $k = 1$ amplituhedra $\mathcal{A}_{n,1,m}$, which are *cyclic polytopes* with n vertices in $\mathbb{P}^m$. Cyclic hyperplane arrangements have been studied by Shannon [22], Ziegler [27], Ramírez Alfonsín [19], and Forge and Ramírez Alfonsín [8]. For an introduction to hyperplane arrangements, see [24].

Remark 5.1. Cyclic polytopes have many faces of each dimension, in the sense of the upper bound theorem of McMullen [17] and Stanley [23]. An analogous property of cyclic hyperplane arrangements is that they have few simplicial faces of each dimension, in the sense of Shannon [22].

Definition 5.2. Let $W \in \mathrm{Gr}_{k+1,n}^{\geq 0}$. By a result of Rietsch [20], W contains a k-dimensional totally positive subspace V. Let $w^{(1)}, \ldots, w^{(k)}$ be a basis of V, and take $w^{(0)} \in W \setminus V$. After replacing $w^{(0)}$ with $-w^{(0)}$ if necessary, we assume that $w^{(0)}$ is *positively oriented* with respect to V, by which we mean that the orthogonal projection of $w^{(0)}$ to $V^{\perp}$ has a positive first component.

We let $\mathcal{H}^W$ be the hyperplane arrangement in $\mathbb{R}^k$ with hyperplanes

$$H_i := \{x \in \mathbb{R}^k : w_i^{(1)} x_1 + \cdots + w_i^{(k)} x_k + w_i^{(0)} = 0\} \text{ for } i \in [n].$$

Then $\mathcal{H}^W$ partitions $\mathbb{R}^k$ into polyhedra, which we call its *faces*. Also, $\mathcal{H}^W$ is *generic*, i.e. the intersection of any $j \leq k$ of its hyperplanes has codimension j, and the intersection of any $j > k$ hyperplanes is empty.

Given $w \in W$, we let $\langle w \rangle \in \mathbb{P}(W)$ denote the line spanned by w. We define the maps

$$\Psi_{\mathcal{H}^W} : \mathbb{R}^k \to \mathbb{P}(W), \qquad x \mapsto \langle x_1 w^{(1)} + \cdots + x_k w^{(k)} + w^{(0)} \rangle,$$
$$\psi_{\mathcal{H}^W} : \mathbb{R}^k \to \{0, +, -\}^n, \quad x \mapsto \operatorname{sign}(x_1 w^{(1)} + \cdots + x_k w^{(k)} + w^{(0)}).$$

The faces of $\mathcal{H}^W$ are precisely the nonempty fibers of $\psi_{\mathcal{H}^W}$. If $\sigma \in \{0, +, -\}^n$ has a nonempty preimage under $\psi_{\mathcal{H}^W}$, we call this fiber the face of $\mathcal{H}^W$ *labeled by* σ. When we identify faces with labels in this way, the face poset of $\mathcal{H}^W$ is an induced subposet of the sign vectors $\{0, +, -\}^n$. We also let $B(\mathcal{H}^W)$ be the subcomplex of bounded faces of $\mathcal{H}^W$.

Remark 5.3. In the literature, a *cyclic hyperplane arrangement* of n hyperplanes in $\mathbb{R}^k$ is usually defined to be an arrangement with hyperplanes

$$H_i := \{x \in \mathbb{R}^k : t_i x_1 + t_i^2 x_2 + \cdots + t_i^k x_k + 1 = 0\} \text{ for } i \in [n],$$

where $0 < t_1 < \cdots < t_n$. These are special cases of our arrangements $\mathcal{H}^W$, since they satisfy the positive orientation condition of Definition 5.2 (see [12, Proposition 6.8]). By Lemma 5.4, our arrangements $\mathcal{H}^W$ are all combinatorially equivalent to each other (i.e. they have the same face poset), which is why we call all of them "cyclic" arrangements.

The main result of this section is that $\mathcal{B}_{n,k,1}(W) \cong B(\mathcal{H}^W)$. The key to the proof is determining the labels of the bounded and unbounded faces of $\mathcal{H}^W$.

Lemma 5.4. *Let $W \in \mathrm{Gr}_{k+1,n}^{>0}$, and $\mathcal{H}^W$ be a cyclic hyperplane arrangement as in Definition 5.2.*
(i) The labels of the bounded faces of $\mathcal{H}^W$ are precisely $\overline{\mathrm{Sign}_{n,k,1}}$.
(ii) The labels of the unbounded faces of $\mathcal{H}^W$ are precisely $\sigma \in \{0, +, -\}^n$ with $\overline{\mathrm{var}}(\sigma) \leq k - 1$.

Theorem 5.5. *In the notation of Definition 5.2, the restriction of $\Psi_{\mathcal{H}^W}$ to $B(\mathcal{H}^W)$ is a homeomorphism from $B(\mathcal{H}^W)$ to $\mathcal{B}_{n,k,1}(W)$, which induces a poset isomorphism on the stratifications of $B(\mathcal{H}^W)$ and $\mathcal{B}_{n,k,1}(W)$. Explicitly, $\Psi_{\mathcal{H}^W}$ sends the face of $\mathcal{H}^W$ labeled by σ to the stratum $\mathcal{B}_\sigma(W)$ of $\mathcal{B}_{n,k,1}(W)$, for all $\sigma \in \overline{\mathrm{Sign}_{n,k,1}}$.*

Dong [6, Theorem 3.1] showed that the bounded complex of a uniform affine oriented matroid (of which the bounded complex of a generic hyperplane arrangement is a special case) is a piecewise linear ball. Therefore Theorem 5.5 implies the following.

Corollary 5.6. *The $m = 1$ amplituhedron is homeomorphic to a closed ball.*

As a further corollary of Theorem 5.5, we obtain the generating function for the stratification of $\mathcal{B}_{n,k,1}(W)$ with respect to dimension, since Buck found the corresponding generating function of $B(\mathcal{H})$ for a generic hyperplane arrangement $\mathcal{H}$ (which only depends on its dimension and the number of hyperplanes).

Corollary 5.7 ([5])**.** *Let $f_{n,k,1}(q) := \sum_{\text{strata } S \text{ of } \mathcal{B}_{n,k,1}(W)} q^{\dim(S)} \in \mathbb{N}[q]$ be the generating function for the stratification of $\mathcal{B}_{n,k,1}(W)$, with respect to dimension. Then*

$$f_{n,k,1}(q) = \sum_{i=0}^{k} \binom{n-k-1+i}{i}\binom{n}{k-i} q^i = \sum_{j=0}^{k} \binom{n-k-1+j}{j}(1+q)^j.$$

For example, we have $f_{5,3,1}(q) = 4q^3 + 15q^2 + 20q + 10$, which we invite the reader to verify from Figure 1. Note that by substituting $q = -1$ into the last expression above, it is easy to check that the Euler characteristic of $\mathcal{B}_{n,k,1}(W)$ equals 1.

We can also use Theorem 5.5 to describe how the cells of the $m = 1$ amplituhedron fit together; see [12, Section 7]. In [12], we also study the image in the $m = 1$ amplituhedron of an arbitrary cell of the totally nonnegative Grassmannian. In particular, we identify which cells are mapped injectively to the $m = 1$ amplituhedron.

Theorem 5.8. *Let D be a* Ⅎ*-diagram of type (k, n). Then the map $S_{M(D)} \to \mathcal{B}_{n,k,1}(W)$, $V \mapsto V^\perp \cap W$ is injective on the cell $S_{M(D)}$ if and only if D satisfies the following conditions: D has at most one $+$ per row, and there is no 0 which has a $+$ above it in the same column and a $+$ elsewhere in the same row. (See [12, Theorem 8.10] for an explicit description of the cells of $\mathcal{B}_{n,k,1}(W)$ in the image of $S_{M(D)}$ in this case.)*

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